

## Part 2: Question 1 (20 points)

Once per day, a tour boat operator brings  $I$  tourists on a boat to an island with the ruins of an old monastery. Upon arriving at the island, the tourists get off the boat and explore the island. Once all of the tourists have finished exploring the island they return to the boat and the operator brings them back to the mainland. Due to legal reasons, the tour boat operator must wait for all tourists to return to the boat and transport all  $I$  tourists back to the mainland together, all at the same time.

Each tourist initially enjoys spending time on the island, but eventually after exploring all of it they become bored and want to return to the mainland. The tourists have heterogeneous preferences for time spent on the island. The derived utility for tourist  $i$  with wealth  $w_i$  from spending  $t \geq 0$  units of time on the island is given by:

$$\phi_i(t, w_i) = \gamma_i t - \frac{1}{2} t^2 + w_i$$

where  $0 < \gamma_1 < \gamma_2 < \dots < \gamma_I$ . Tourist  $i$  does not have complete control over  $t$ . The boat only returns to the mainland once all  $I$  tourists have returned to the boat. Therefore if each tourist  $i$  chooses to spend  $t_i$  units of time on the island, the boat will only leave after  $t = \max\{t_1, t_2, \dots, t_I\}$  units of time.

- (i) [5 points] Show that if each tourist individually decides how long to spend on the island (not taking into account their impact on other tourists), the boat will spend  $t^* = \gamma_I$  units of time on the island.
- (ii) [5 points] Show that a social planner who values each consumer's utility equally will choose that the boat spends  $t^\circ = \frac{1}{I} \sum_{i=1}^I \gamma_i$  units of time on the island.
- (iii) [3 points] Give an intuitive reason for why  $t^* \neq t^\circ$ .
- (iv) [7 points] In order to correct for this welfare loss, the tour boat operator decides to charge each tourist a personalized price  $p_i$  for each unit of time spent on the island. The prices are chosen such that all tourists choose to return to the boat after  $t^\circ$  units of time. The price  $p_i$  may be negative for some consumers, in which case the tour boat operator pays the consumer for each unit of time spent on the island. Find a set of prices  $p_i$  and lump-sum transfers  $T_i$  for each consumer such that each consumer receives the same utility as in the planner's problem in part (ii).

*Solution:*

- (i) Only one of the tourists will be pivotal in how long  $t$  is. This is the length of time the tourist who individually wants to spend the longest time on the island. If alone on the island, each tourist  $i$  solves the following problem:

$$\max_{t_i \geq 0} \gamma_i t_i - \frac{1}{2} t_i^2$$

This has a first-order condition  $\gamma_i - t_i \leq 0$ , with equality if  $\gamma_i > 0$ , which holds for all  $i$ . Therefore, if alone on the island, consumer  $i$  would choose  $t_i = \gamma_i$ . Because the boat leaves when the last tourist comes back to the boat, the boat will leave after  $t^* = \max\{\gamma_1, \dots, \gamma_I\} = \gamma_I$  units of time.

- (ii) A planner chooses  $t$  to maximize the welfare of all tourists. They choose  $t$  to satisfy:

$$\max_{t \geq 0} \sum_{i=1}^I \left( \gamma_i t - \frac{1}{2} t^2 \right)$$

The first-order condition is  $\sum_{i=1}^I \gamma_i - It \leq 0$  with equality if  $\sum_{i=1}^I \gamma_i > 0$  which holds in our case. So the planner chooses  $t^\circ = \frac{1}{I} \sum_{i=1}^I \gamma_i$ , the average of what each individual would choose if they were going to the island alone.

- (iii) The  $I$ th tourist has an externality on all of the others as the utility of tourists 1 to  $I-1$  depends  $I$ 's actions. Tourist  $I$  does not internalize this externality and the competitive outcome differs from the social planner's solution (unless the planner places 100% of the social weights on tourist  $I$ ).
- (iv) For each consumer  $i$  we need to find a price  $p_i$  per unit of time spent on the island such that they will choose  $t^\circ = \frac{1}{I} \sum_{i=1}^I \gamma_i$ . The price should be equal to  $p_i = \gamma_i - \frac{1}{I} \sum_{i=1}^I \gamma_i$ . To see this, with this price, consumer  $i$ 's problem becomes:

$$\max_{t \geq 0} \gamma_i t - \frac{1}{2} t^2 - \underbrace{\left( \gamma_i - \frac{1}{I} \sum_{i=1}^I \gamma_i \right)}_{=p_i} t$$

The first-order condition is  $-t + \frac{1}{I} \sum_{i=1}^I \gamma_i = 0$  which has the solution  $t = \frac{1}{I} \sum_{i=1}^I \gamma_i$ , the same as the social planner's problem. Consumers that enjoy spending longer on the island (have an above-average  $\gamma_i$ ) pay a tax, while those that prefer spending less time on the island are subsidized to explore it longer.

This solution involves each consumer effectively paying a tax (or receiving a subsidy)

of:

$$p_i \times t^o = \left( \gamma_i - \frac{1}{I} \sum_{i=1}^I \gamma_i \right) \times \left( \frac{1}{I} \sum_{i=1}^I \gamma_i \right)$$

If we transfer each consumer  $i$  back exactly  $\left( \gamma_i - \frac{1}{I} \sum_{i=1}^I \gamma_i \right) \times \left( \frac{1}{I} \sum_{i=1}^I \gamma_i \right)$  (or charge a lump-sum tax if  $p_i < 0$ ), then they are equally well off as under the planner's problem. This is also budget balancing as each consumer pays the same tax and receives the same amount back in the form of a transfer.

## Part 2: Question 2 (20 Points)

Consider a pure exchange economy with  $I > 1$  consumers and  $L > 1$  goods. Each consumer  $i$ 's preferences are complete, transitive, continuous, strictly convex and strongly monotone. Each consumer's consumption set is  $X_i = \mathbb{R}_+^L$ . There is a single firm whose only production technology is free disposal:  $\mathcal{Y}_1 = \{\mathbf{y}_1 \in \mathbb{R}^L : \mathbf{y}_1 \leq \mathbf{0}\}$ . The aggregate endowment vector satisfies  $\bar{\omega} \gg \mathbf{0}$  (strictly positive in all goods).

In this economy there is an allocation  $(\mathbf{x}_1^*, \dots, \mathbf{x}_I^*, \mathbf{y}_1^*)$  with  $\mathbf{x}_i^* \gg \mathbf{0}$  for all  $i$  and  $\mathbf{y}_1^* = \mathbf{0}$  that is Pareto optimal.

In this question, we want to prove that if  $\omega_i = \mathbf{x}_i^*$  for all  $i$ , then  $(\mathbf{x}_1^*, \dots, \mathbf{x}_I^*, \mathbf{y}_1^*)$  with  $\mathbf{y}_1^* = \mathbf{0}$  is the unique Walrasian equilibrium in this economy.

To prove this, we split it into a number of steps.

- (i) [**3 points**] Based on the assumptions above, we are guaranteed that an equilibrium exists in this economy (using the existence proof from class). Call the allocation in this equilibrium  $(\mathbf{x}'_1, \dots, \mathbf{x}'_I, \mathbf{y}'_1)$  and the price vector  $\mathbf{p}'$ . Show that  $\mathbf{p}' \gg \mathbf{0}$  (i.e. argue that we cannot have any  $p_\ell = 0$ ).
- (ii) [**3 points**] Using (i), show that  $\mathbf{y}'_1 = \mathbf{0}$ .
- (iii) [**3 points**] Show that  $\mathbf{x}'_i \sim_i \mathbf{x}_i^*$  for all  $i$ . *Hint:* Use the fact that  $\omega_i = \mathbf{x}_i^*$  for all  $i$  and that  $(\mathbf{x}_1^*, \dots, \mathbf{x}_I^*, \mathbf{y}_1^*)$  is Pareto optimal.
- (iv) [**3 points**] Use (ii) and (iii) above to show that  $(\mathbf{x}_1^*, \dots, \mathbf{x}_I^*, \mathbf{y}_1^*)$  with price vector  $\mathbf{p}'$  is a Walrasian equilibrium.
- (v) [**8 points**] Using the fact that  $(\mathbf{x}_1^*, \dots, \mathbf{x}_I^*, \mathbf{y}_1^*)$  is a Walrasian equilibrium allocation from part (iv), prove that this equilibrium allocation is unique. *Hint:* Use proof by contradiction. Suppose there is another allocation that is also an equilibrium, and make use of the strict convexity of preferences.

*Solution:*

- (i) In any equilibrium with strongly monotone preferences we must have  $\mathbf{p} \gg \mathbf{0}$ . Otherwise consumers would demand an infinite quantity of the good with a zero price, which would violate market clearing.
- (ii) Because all prices are positive, the profit-maximizing production plan for the firm whose only technology is disposal is to produce zero output which earns zero profits. Any negative output would incur losses.
- (iii) Because it is chosen in equilibrium,  $\mathbf{x}'_i \succeq_i \boldsymbol{\omega}_i = \mathbf{x}^*_i$  for all  $i$ . Moreover, the equilibrium allocation is feasible. But because the allocation  $(\mathbf{x}^*_1, \dots, \mathbf{x}^*_I, \mathbf{y}^*_1)$  is Pareto optimal, there does not exist any other feasible allocation where  $\mathbf{x}'_i \succeq_i \mathbf{x}^*_i$  for all  $i$  with strict preference for at least one  $i$ . Therefore no consumer strictly prefers  $\mathbf{x}'_i$  and so  $\mathbf{x}'_i \sim_i \mathbf{x}^*_i$  for all  $i$ .
- (iv) Because  $\mathbf{x}'_i$  is an optimal bundle for consumer  $i$  at prices  $\mathbf{p}'$  (because it is an equilibrium) and because  $\mathbf{x}'_i \sim_i \mathbf{x}^*_i$  (from part iii), then  $\mathbf{x}^*_i$  is also an optimal bundle. Moreover,  $\mathbf{y}^*_1 = \mathbf{y}'_1 = \mathbf{0}$  is also optimal at prices  $\mathbf{p}'$  for the firm.
- (v) Suppose toward a contradiction there was another equilibrium with allocation  $\mathbf{x}'_i \neq \boldsymbol{\omega}_i$  for at least one  $i$ . Because  $\mathbf{x}'$  is an equilibrium,  $\mathbf{x}'_i \succeq_i \boldsymbol{\omega}_i \forall i$ . It also satisfies feasibility:  $\sum_{i=1}^I \mathbf{x}'_i = \sum_{i=1}^I \boldsymbol{\omega}_i$ . By strict convexity, for the  $i$  where  $\mathbf{x}'_i \neq \boldsymbol{\omega}_i$ ,  $\mathbf{x}''_i = \frac{1}{2}\mathbf{x}'_i + \frac{1}{2}\boldsymbol{\omega}_i$  satisfies  $\mathbf{x}''_i \succ_i \boldsymbol{\omega}'_i$ . Moreover,  $\mathbf{x}''_i$  is feasible because:

$$\sum_{i=1}^I \mathbf{x}''_i = \frac{1}{2} \sum_{i=1}^I \mathbf{x}'_i + \frac{1}{2} \sum_{i=1}^I \boldsymbol{\omega}_i = \sum_{i=1}^I \boldsymbol{\omega}_i$$

So  $\mathbf{x}''$  Pareto dominates  $\{\boldsymbol{\omega}_i\}_{i=1}^I$  (because it is feasible, weakly preferred for all  $i$  and strictly preferred for at least one). But this contradicts that it was Pareto optimal. Therefore there is only one equilibrium allocation from this initial endowment.